

Black p-Branes versus Black Holes in Non-asymptotically flat Einstein-Yang-Mills theory

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We present a class of non-asymptotically flat (NAF) charged black p-branes (BpB) with p-compact dimensions in higher dimensional Einstein-Yang-Mills (EYM) theory. Asymptotically the NAF structure manifests itself as an anti-de-sitter(AdS) spacetime. By comparing the entropies of BpB with those of black holes (BH) in same dimensions we derive transition criteria between the two types of black objects. An event horizon (r_+) versus charge square (Q^2) plot reveals such a transition provided both are characterized by the same r_+ and Q .

I. INTRODUCTION

It is well-known that by uplifting d -dimensional dilatonic black holes (BH) one obtains $d + p$ -dimensional black p -branes (BpB) with extended event horizons [1]. A black string (BS), for $p = 1$ is an extension with one extra dimension whose horizon has a product topology such as $R \times S^{d-2}$. Naturally the simplest member of this class constitutes of the chargeless Schwarzschild metric in $d = 4$ and $p = 1$, so that the topology becomes $R \times S^2$ [2]. In case that the extra dimension is compact the end points may be identified to give a BS with horizon topology $S^1 \times S^2$. Concerning BS (and more generally BpB) an interesting problem that gave birth to a considerable literature in recent years is their instability against decay into BH (or vice versa). This problem was pointed out first, through perturbation analysis by Gregory and Laflamme (GL) which came to be known as GL instability [3]. Such a perturbative stability / instability applies to asymptotically flat (AF) metrics, which should not be reliable for non-asymptotically flat (NAF) spacetimes. Both for magnetic [4] and electric charges [5] the GL instability has been shown perturbatively in AF spacetimes to remain intact.

In this Letter we employ local thermodynamical stability / instability arguments supplemented with entropy comparison to relate charged BpB and BH in NAF spacetimes. In doing this we assume that both, the charges and event horizon radii of BpB and BH are same. Our system consists of d -dimensional Einstein-Yang-Mills (EYM) theory in a compact p -dimensional brane world which admits NAF black objects. Among a large class of BH solutions which can be used to generate a family of BpB we choose a specific BH solution so that technically it becomes tractable. In other words, as long as the NAF condition is assumed the freedom of alternative solutions is always available. Non-Abelian gauge fields were considered as BS solutions by other researchers [6]. Our approach, however, different from other studies, presents exact non-Abelian solutions in all dimensions. Asymptotically our solutions represent anti-de Sitter (AdS) spacetimes. The heat capacity of our solution diverges for the critical condition $d = 2(p + 1)$, where $d + p$ and p refer to the spacetime and brane dimensions, respectively. We resort next to compare the entropies of BpB and BH with equal event horizons r_+ and charges Q . From comparison of entropy expressions we plot r_+ versus Q^2 for $d + p = 10$ to identify the regions of both BpB and BH. The intersecting curve determines naturally the transition between BpB to BH and vice versa. Fig. 1 implies that for a given dimension $d + p = \text{constant}$, increasing p / decreasing d , favors a larger region for BH/ BpB. Organization of the Letter is as follows.

In section II we present our exact EYM solutions in all dimensions. Thermodynamic stability of the solution is discussed in section III. We summarize our results in Conclusion which appears in section IV.

II. THE NAF EYM SOLUTION

Our $d + p$ -dimensional action with p -compact dimensions in EYM theory is given by

$$I = \frac{1}{16\pi G_{(d+p)}} \int_0^{L_p} dz_p \dots \int_0^{L_1} dz_1 \int d^d x \sqrt{-g} (R - \mathcal{F}) \quad (1)$$

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in which

$$\mathcal{F} = \text{Tr}(F_{\mu\nu}^{(a)} F^{(a)\mu\nu}) \quad (2)$$

and

$$\mathbf{F}^{(a)} = \mathbf{d}\mathbf{A}^{(a)} + \frac{1}{2\sigma} C_{(b)(c)}^{(a)} \mathbf{A}^{(b)} \wedge \mathbf{A}^{(c)}, \quad (3)$$

is the YM 2-form field with structure constants $C_{(b)(c)}^{(a)}$. Here R is the Ricci scalar, the coupling constant σ is expressed in terms of the YM charge and

$$\text{Tr}(\cdot) = \sum_{a=1}^{\frac{(d-1)(d-2)}{2}} (\cdot). \quad (4)$$

Note that $G_{(d+p)}^{BpB}$ represents the $d+p$ -dimensional Newton constant while L_i ($1 \leq i \leq p$) stands for a set of compact dimensions. We note also that $L_i = L$, for all $1 \leq i \leq p$, need not be assumed here. For future reference we prefer to use $L_i \neq L_j$ for $i \neq j$ in this paper. As a matter of fact our results in this paper will be valid irrespective of the compact volume. Our ultimate choice in this study will be $L_1 \dots L_p = 1$, so that if $L_i = \epsilon$ and $L_j = \frac{1}{\epsilon}$, for some i and j and $\epsilon \rightarrow 0$ we preserve the same volume with non-compact translational symmetry. Our pure magnetic YM potential follows from the higher dimensional version [7] of Wu-Yang ansatz which is given by

$$\begin{aligned} \mathbf{A}^{(a)} &= \frac{Q}{r^2} C_{(i)(j)}^{(a)} x^i dx^j, \quad Q = \text{YM magnetic charge}, \quad r^2 = \sum_{i=1}^{d-1} x_i^2, \\ 2 \leq j+1 \leq i \leq d-1, \quad \text{and} \quad 1 \leq a \leq (d-1)(d-2)/2, \\ x_1 &= r \cos \theta_{d-3} \sin \theta_{d-4} \dots \sin \theta_1, \quad x_2 = r \sin \theta_{d-3} \sin \theta_{d-4} \dots \sin \theta_1, \\ x_3 &= r \cos \theta_{d-4} \sin \theta_{d-5} \dots \sin \theta_1, \quad x_4 = r \sin \theta_{d-4} \sin \theta_{d-5} \dots \sin \theta_1, \\ &\dots \\ x_{d-2} &= r \cos \theta_1. \end{aligned} \quad (5)$$

Our choice for the BpB metric ansatz is given by

$$\begin{aligned} ds^2 &= e^{-b\psi} \left(-f(r) dt^2 + \frac{dr^2}{f(r)} + h(r)^2 d\Omega_{(d-2)}^2 \right) + e^{\frac{b(d-2)}{p}\psi} dz^i dz^i, \\ (i &= 1, 2, \dots, p) \end{aligned} \quad (6)$$

in which $\psi(r)$, $f(r)$ and $h(r)$ are some functions of r to be found and $d\Omega_{(d-2)}^2$ is the $d-2$ -dimensional unit spherical line element. Variation of the action with respect to $g_{\mu\nu}$ yields

$$G_\mu^\nu = T_\mu^\nu, \quad (7)$$

where

$$T_\mu^\nu = 2 \left[\text{Tr} \left(F_{\mu\lambda}^{(a)} F^{(a)\nu\lambda} \right) - \frac{1}{4} \mathcal{F} \delta_\mu^\nu \right] \quad (8)$$

or explicitly

$$T_\mu^\nu = -\frac{(d-3)(d-2)Q^2}{2h^4} e^{2b\psi} \text{diag} \left[1, 1, \overbrace{\kappa, \kappa, \dots, \kappa}^{d-2\text{-times}}, \overbrace{1, 1, \dots, 1}^{p\text{-times}} \right], \quad \kappa = \frac{d-6}{d-2} \quad (9)$$

and non-zero G_μ^ν are given by

$$G_t^t = \frac{(d-2)e^{b\psi}}{2h^2} \left[\frac{d+p-2}{4p} b^2 f h^2 \psi'^2 + h h' f' + 2f h h'' + (d-3)(f h'^2 - 1) \right], \quad (10)$$

$$G_r^r = \frac{(d-2)e^{b\psi}}{2h^2} \left[-\frac{d+p-2}{4p} b^2 f h^2 \psi'^2 + h h' f' + (d-3)(f h'^2 - 1) \right], \quad (11)$$

$$G_{\theta_i}^{\theta_i} = \frac{e^{b\psi}}{h^2} \left[\frac{d+p-2}{4p} \left(\frac{d-2}{2} b^2 f h^2 \psi'^2 \right) + (d-3) h (h' f)' + \frac{(d-3)(d-4)}{2} (f h'^2 - 1) + \frac{1}{2} h^2 f'' \right], \quad (12)$$

$$G_{z_i}^{z_i} = \frac{(d-2)e^{b\psi}}{2h^2} \left[\frac{d+p-2}{4p} \left(b^2 f h^2 \psi'^2 - \frac{4b(f\psi')' h^2}{(d-2)} - 4h f b \psi' h' \right) + \frac{h^2 f''}{d-2} + 2h (h' f)' + (d-3)(f h'^2 - 1) \right]. \quad (13)$$

The YM equations also follow from the action as

$$\mathbf{d} \left(\star \mathbf{F}^{(a)} \right) + \frac{1}{\sigma} C_{(b)(c)}^{(a)} \mathbf{A}^{(b)} \wedge \star \mathbf{F}^{(c)} = 0 \quad (14)$$

where the hodge star \star implies duality. By direct substitution, one can show that YM equations are satisfied. From $T_t^t = T_r^r$ (or $G_t^t = G_r^r$) it follows that

$$\frac{(d+p-2)b^2}{4p} \psi'^2 = -\frac{h''}{h}, \quad (15)$$

which after setting $h = \xi e^{\alpha\psi}$ we obtain

$$\psi = \frac{\alpha}{\alpha^2 + \left(\frac{d+p-2}{4p}\right)b^2} \ln(\xi_1 r + \xi_2). \quad (16)$$

Here α and ξ are two parameters to be fixed while ξ_1 and ξ_2 are two integration constants. By shifting and rescaling r coordinate we set them as $\xi_1 = 1$ and $\xi_2 = 0$ which gives

$$\psi = \frac{\alpha \ln r}{\alpha^2 + \left(\frac{d+p-2}{4p}\right)b^2}. \quad (17)$$

Now, we substitute ψ and h into the EYM BpB equations (7) and after the choice $\alpha = \frac{b}{2}$, the field equations admit

$$f(r) = \Xi \left(1 - \left(\frac{r_+}{r} \right)^{\frac{(p+1)(d-2)}{d+2p-2}} \right) r^{\frac{2(d+p-2)}{d+2p-2}}, \quad (18)$$

in which

$$\Xi = \frac{(d-3)(d+p-2)}{Q^2(d-2)(p+1)} \quad (19)$$

and

$$\xi^2 = \frac{Q^2(d+2p-2)}{d+p-2}. \quad (20)$$

Here r_+ is an integration constant which represents the radius of the event horizon. Upon rewriting the line element in the form

$$ds^2 = -f_1(r) dt^2 + \frac{dr^2}{f_2(r)} + f_3(r) dz_i dz^i + f_4(r) d\Omega_{(d-2)}^2 \quad (21)$$

the metric functions take the following forms

$$f_1(r) = f(r) e^{-b\psi} = \Xi \left(1 - \left(\frac{r_+}{r} \right)^{\frac{(p+1)(d-2)}{d+2p-2}} \right) r^{\frac{2(d-2)}{d+2p-2}}, \quad (22)$$

$$f_2(r) = f(r) e^{b\psi} = \Xi \left(1 - \left(\frac{r_+}{r} \right)^{\frac{(p+1)(d-2)}{d+2p-2}} \right) r^2, \quad (23)$$

$$f_3(r) = e^{\frac{b(d-2)}{p}\psi} = r^{\frac{2(d-2)}{d+2p-2}}, \quad f_4(r) = e^{-b\psi} h(r)^2 = \xi^2. \quad (24)$$

In order to understand the physical implication of this class of NAF solutions we investigate their asymptotic behaviors for $r \gg r_+$ and for $r = r_+ + \epsilon$ ($\epsilon^2 \approx 0$). For this purpose we make the transformation

$$\begin{aligned} r &\rightarrow r^{-k} \\ (k &= \frac{d+2p-2}{d-2}), \end{aligned} \quad (25)$$

followed by the scalings

$$\begin{aligned} t &\rightarrow \left(\frac{\Xi}{k}\right) t, \\ z_i &\rightarrow \left(\frac{\sqrt{\Xi}}{k}\right) z_i \end{aligned} \quad (26)$$

to yield for $r \gg r_+$

$$ds^2 \simeq \frac{1}{r^2} (-dt^2 + dr^2 + dz_i dz^i) + \frac{d-3}{k(p+1)} d\Omega_{(d-2)}^2. \quad (27)$$

This is the geometry of the $(p+2)$ dimensional anti-de Sitter (AdS) spacetime times the $(d-2)$ dimensional sphere (i.e. $adS_{p+2} \times S^{d-2}$). In analogy, near the horizon (i.e. $r = r_+ + \epsilon$, $\epsilon > 0$, $\epsilon^2 \approx 0$) we have our line element

$$ds^2 = -C_0 dt^2 + C_1 dr^2 + C_2 (dz_i)^2 + \xi^2 d\Omega_{d-2}^2 \quad (28)$$

for appropriate constants C_i . This is the product space of 2-dimensional Minkowski space with p -torus and $(d-2)$ dimensional sphere (i.e. $M^2 \times T^p \times S^{d-2}$).

The simplest member in this class of solutions is given by the choice $d = 4, p = 1$. This yields the line element of a black string

$$ds^2 = -\frac{3(r-r_+)}{4Q^2} dt^2 + \frac{4Q^2}{3r(r-r_+)} dr^2 + r dz^2 + \frac{4Q^2}{3} d\Omega_2^2 \quad (29)$$

which is non-singular and manifestly NAF. From our foregoing argument this asymptotes (for $r \gg r_+$) to the spacetime $adS_3 \times S^2$ while for $r \approx r_+$ it is $M^2 \times S^1 \times S^2$. We note that our solutions are generically regular for $p = 1$, and singular at $r = 0$ for $p > 1$. The singularity shows itself in the Kretschmann scalar while other scalars are all regular.

III. THERMODYNAMICS STABILITY

The entropy of the BpB metric (21) is defined by

$$S_{BpB} = \frac{A_H}{4G_{(d+p)}^{BpB}}, \quad (30)$$

in which $G_{(d+p)}^{BpB} = G_d \prod_{k=1}^p L_k$ while

$$A_H = \frac{2\pi^{\frac{d-1}{2}} \prod_{k=1}^p L_k}{\Gamma\left(\frac{d-1}{2}\right)} f_4(r_+)^{\frac{d-2}{2}} f_3(r_+)^{\frac{p}{2}}. \quad (31)$$

Here we set $16\pi G_{(d)} = 1$ and therefore

$$S_{BpB} = \frac{8\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} f_4(r_+)^{\frac{d-2}{2}} f_3(r_+)^{\frac{p}{2}}. \quad (32)$$

which, upon substitution from above, implies

$$S_{BpB} = \frac{8\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{Q^2 (d+2p-2)}{d+p-2} \right)^{\frac{d-2}{2}} r_+^{\frac{p(d-2)}{d+2p-2}}. \quad (33)$$

Now, the Hawking temperature T_H and specific heat capacity C_Q of the BpB are given by

$$T_H = \frac{1}{4\pi} f'_1(r_+) = \frac{(d+p-2)(d-3)}{4\pi(d+2p-2)} \frac{d-2p-2}{Q^2 r_+} \quad (34)$$

and

$$C_Q = T_H \left(\frac{\partial S_{BpB}}{\partial T_H} \right)_Q = \frac{p(d-2)}{d-2p-2} S_{BpB}. \quad (35)$$

Since, from (33) $S_{BpB} > 0$, the only criticality condition that we may obtain is from $d = 2(p+1)$. For $d > 2(p+1)$, it yields $C_Q > 0$, so that we propose instability criteria which turn out to be a relation between dimensions in the present case. Our aim next, is to compare the entropy of $d+p$ -dimensional non-asymptotically flat (NAF) EYM BpB with the entropy of the $d+p$ -dimensional (NAF) EYMBH whose metric is given by

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + h^2 d\Omega_{(d+p-2)}^2 \quad (36)$$

where h is a constant to be fixed. The corresponding action for the $d+p$ -dimensional EYM theory is also given by

$$I_{BH}^{(d+p)} = \frac{1}{16\pi G_{(d+p)}^{BH}} \int d^{d+p}x \sqrt{-g} (R - \mathcal{F}) \quad (37)$$

in which \mathcal{F} is the YM invariant (2). The YM equations are satisfied with Einstein tensor components as

$$G_\mu^\nu = \left[-\frac{(d+p-3)(d+p-2)}{2h^2}, -\frac{(d+p-3)(d+p-2)}{2h^2}, -\frac{(d+p-3)(d+p-4) + f''h^2}{2h^2}, \dots \right]. \quad (38)$$

The energy-momentum tensor components follow from (9) with $b = 0$ and $d \rightarrow d+p$, i.e.,

$$T_\mu^\nu = -\frac{(d+p-3)(d+p-2)Q_{BH}^2}{2h^4} \text{diag} \left[1, 1, \overbrace{\kappa, \kappa, \dots, \kappa}^{d+p-2 \text{ times}} \right], \quad \kappa = \frac{d+p-6}{d+p-2}. \quad (39)$$

The Einstein equations (7) imply now that

$$h^2 = Q_{BH}^2 \quad (40)$$

with

$$f = \frac{2(d+p-3)}{Q_{BH}^2} r^2 + C_1 r + C_2, \quad (41)$$

where Q_{BH} stands for the charge of the BH. Here C_1 and C_2 are two integration constants which for our purpose we set $C_1 = 0$ to cast f into the form

$$f = \frac{2(d+p-3)}{Q_{BH}^2} r^2 \left(1 - \left(\frac{r_h}{r} \right)^2 \right) \quad (42)$$

where r_h indicates the horizon of the black hole. The entropy of NAF-EYMBH is given by

$$S_{BH} = \frac{A_H}{4G_{(d+p)}^{BH}} = \frac{8\pi^{\frac{d+p+1}{2}}}{\Gamma\left(\frac{d+p-1}{2}\right)} (Q_{BH}^2)^{\frac{(d+p-2)}{2}}, \quad (43)$$

in which for $d+p$ dimensional BH we have used $16\pi G_{(d+p)}^{BH} = 1$ (i.e. the volume of the extra space due to branes is chosen as $L_1 L_2 \dots L_p = 1$). Our final argument is to define the micro-canonical equilibrium condition for the EYM BpB as $S_{BpB} \geq S_{BH}$ i.e.,

$$\frac{\Gamma\left(\frac{d+p-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{d+2p-2}{d+p-2} \right)^{\frac{d-2}{2}} (Q_{BpB}^2)^{\frac{d-2}{2}} r_+^{\frac{p(d-2)}{d+2p-2}} \geq \pi^{\frac{p}{2}} (Q_{BH}^2)^{\frac{(d+p-2)}{2}}. \quad (44)$$

This inequality can be interpreted in two ways. First, we consider that a transition from our BpB to the BH is prohibited provided

$$Q_{BH}^2 \leq \left(\frac{\Gamma\left(\frac{d+p-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right) \pi^{\frac{p}{2}}} \left(\frac{d+2p-2}{d+p-2} \right)^{\frac{d-2}{2}} (Q_{BpB}^2)^{\frac{d-2}{2}} r_+^{\frac{p(d-2)}{d+2p-2}} \right)^{\frac{2}{(d+p-2)}}, \quad (45)$$

which defines a maximum charge for the resultant black hole to which the BpB can not decay.

The second interpretation is to assume $Q_{BH}^2 = Q_{BpB}^2 = Q^2$ which leads us to the corresponding critical curve defined by $S_{BpB} = S_{BH}$. This implies that

$$r_+ = \left(\frac{\Gamma\left(\frac{d-1}{2}\right) (\pi Q^2)^{\frac{p}{2}}}{\Gamma\left(\frac{d+p-1}{2}\right)} \left(\frac{d+p-2}{d+2p-2} \right)^{\frac{d-2}{2}} \right)^{\frac{d+2p-2}{p(d-2)}}. \quad (46)$$

In Fig. 1 we plot r_+ versus Q^2 for $d+p = 10$. This displays the critical curves showing the possible regions for BH versus BpB.

IV. CONCLUSION

In NAF, EYM theory we obtained for a particular class of solutions a critical boundary curve that separates BpB from BH bearing equal horizon radii r_+ and charges Q . We argue that our stability treatment doesn't depend on the particular solution but is more general. The NAF character manifests itself asymptotically as an AdS spacetime with a suitable effective cosmological constant. The critical curve arises from entropy comparison for the two types of black objects. It reflects the relative weight of dimensions p and d for each given case $d+p \geq 5$. In particular we plotted r_+ versus Q^2 to represent the case of $d+p = 10$. We found also that the specific heat function diverges for the critical condition $d = 2(p+1)$ which signals a possible phase change. Such critical cases arise, for instance, for the dimensions $d+p = 5, 8, 11, \dots$. At these particular dimensions the entropy difference remains still a reliable test to check possible transition from BpB to BH and vice versa. As a final remark, we admit that our method applies only for charged black objects which doesn't work for neutral cases.

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Figure Caption:

As a result of the entropy argument for the particular case $d+p = 10$, we obtain this informative plot of common effective horizon radius r_+ versus common charge square Q^2 . Each curve represents the critical boundary between a BpB and a BH. Comparison of entropies (from Eq. 39) suggests that the left (or up) of each curve represents a BpB while the right (or down) of each curve favors the BH state. For a constant r_+ it is observed that increment in charge transforms a BpB into a BH. Conversely, for a fixed charge, increasing the horizon radius r_+ goes toward BpB from the BH state.

This figure "FIG01.jpg" is available in "jpg" format from:

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